# Simultaneous linear equations

Let us consider the simultaneous equations be

$$\phi_1(D)x + \psi_1(D)y = F_1(t) \quad \text{—— (1)}$$

$$\phi_2(D)x + \psi_2(D)y = F_2(t) \quad \text{—— (2)}$$

Here $x$ and $y$ are dependent variables and $t$ is an independent variable.

To solve these, we first eliminate one of the dependent variables $x$ or $y$ from

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these equations and then we get one differential equation connecting one dependent and one indepent variable. Then we solve it as usual and get one dependent variable in term of the independent variable. Then using this dependent variable already obtained in the equⁿ ① and ②, the other dependent variable can be found out.

<u>Other method</u> :- When two simultaneous linear equations of first order in $x$ and $y$ be given, then we see that these two equations involve $x, y, \frac{dx}{dt}$, and $\frac{dy}{dt}$. If we ~~not~~ now differentiate these two equations once with respect to the independent variable $t$, we get four equations involving $x, y, \frac{dx}{dt}, \frac{dy}{dt}, \frac{d^2x}{dt^2}$ and $\frac{d^2y}{dt^2}$. Now eliminating $y, \frac{dy}{dt}$, and $\frac{d^2y}{dt^2}$ from these four equations we get a 2nd order differential equation in $x$ and solving it we get $x$ as a function of $t$. Then we proceed as in the previous case.

<u>Q.</u> Solve
$$\frac{dx}{dt} - 7x + y = 0$$
$$\frac{dy}{dt} - 2x - 5y = 0$$

<u>Solution</u>:- using the symbol $D$ for $\frac{d}{dt}$, the given equations can be written as

$(D-7)x + y = 0$ —— (1)

$-2x + (D-5)y = 0$ —— (2)

eliminating y between (1) and (2) we get

$\{(D-5)(D-7)+2\}x = 0$

$\Rightarrow (D^2 - 12D + 37)x = 0$

The auxiliary equation is $m^2 - 12m + 37 = 0$.

$\therefore$ we get $m = 6 \pm i$

$\therefore x = e^{6t}(c_1 \cos t + c_2 \sin t)$ —— (3)

where $c_1$ and $c_2$ being arbitrary constants.

then we have

$$\frac{dx}{dt} = e^{6t}\left[(-c_1 \sin t + c_2 \cos t)\right] + 6e^{6t}(c_1 \cos t + c_2 \sin t) \quad (4)$$

substituting for $x$ and $\frac{dx}{dt}$ in the first eqn.

we get

$$y = c_1 e^{6t} \cos t + c_2 e^{6t} \sin t + c_1 e^{6t} \sin t - c_2 e^{6t} \cos t$$

$$= e^{6t}\left[(c_1 - c_2)\cos t + (c_1 + c_2)\sin t\right] \quad (5)$$

Hence the complete solution is given by (3) and (5).

Q. Solve $\frac{dx}{dt} + 5x + y = e^t$

$\frac{dy}{dt} - x + 3y = e^{2t}$.

Solⁿ :- From the first equⁿ we have

$$y = e^t - \frac{dx}{dt} - 5x \quad \text{--- (1)}$$

putting (1) in the 2nd equⁿ we get

$$\frac{d^2x}{dt^2} + 8\frac{dx}{dt} + 16x = 4e^t - e^{2t}$$

which is a linear equⁿ of 2nd order in x.

Its ~~Comp~~ A.E $m^2 + 8m + 16 = 0$

$$\Rightarrow (m+4)^2 = 0 \Rightarrow m = -4, -4$$

Its C.F is $(c_1 + c_2 t)e^{-4t}$

where $c_1$ and $c_2$ being arbitrary constants.

The particular integral is

$$\frac{1}{(D+4)^2} \cdot 4e^t - \frac{1}{(D+4)^2} \cdot e^{2t}$$

$$= \frac{4}{25}e^t - \frac{1}{36}e^{2t}$$

$$\therefore x = (c_1 + c_2 t)e^{-4t} + \frac{4}{25}e^t - \frac{1}{36}e^{2t} \quad \text{--- (2)}$$

Hence from (1) we get

$$y = -(c_1 + c_2 + c_2 t)e^{-4t} + \frac{1}{25}e^t + \frac{7}{36}e^{2t} \quad \text{--- (3)}$$

(2) and (3) constitute the solution of the given equations.

Q. $\frac{d^2x}{dt^2} - 3x - 4y = 0$

$\frac{d^2y}{dt^2} + x + y = 0$

Solⁿ:- From the 2nd equation we have,

$$x = -\frac{d^2y}{dt^2} - y \quad \text{--- (1)}$$

putting (1) in the first equation we get

$$\frac{d^4y}{dx^4} - 2\frac{d^2y}{dt^2} + y = 0$$

A.E is $m^4 - 2m^2 + 1 = 0$

$\Rightarrow (m^2-1)^2 = 0$

$\Rightarrow m = \pm 1, \pm 1 = +1, +1, -1, -1$

$\therefore$ C.F $y = (c_1 + c_2 t)e^t + (c_3 + c_4 t)e^{-t}$ --- (2)

where $c_1, c_2, c_3, c_4$ being arbitrary consts.

Hence from (1) we get,

$$x = -2(c_1 + c_2 + c_2 t)e^t - 2(c_3 - c_4 + c_4 t)e^{-t} \quad \text{--- (3)}$$

(2) and (3) constitute the solution of the given equations.

H-W solve $\frac{dx}{dt} + 4x + 3y = t$

$\frac{dy}{dt} + 2x + 5y = e^t$.

Solve

$$\frac{dx}{dt} = 2y ; \frac{dy}{dt} = 2z ; \frac{dz}{dt} = 2x.$$

Solution:-

Here $\frac{dx}{dt} = 2y$ —①

$\frac{dy}{dt} = 2z$ —②

$\frac{dz}{dt} = 2x$ —③

Differentiate ① w.r.t.o t we get

$$\frac{d^2x}{dt^2} = 2\frac{dy}{dt} = 4z \text{ from ②}$$

Again differenting w.r.to t we get

$$\frac{d^3x}{dt^3} = 4\frac{dz}{dt} = 4\times 2x$$

$$\Rightarrow \frac{d^3x}{dt^3} = 8x$$

$$\Rightarrow \frac{d^3x}{dt^3} - 8x = 0 \quad \text{—④}$$

Which is a linear equ$^n$ in $x$.

A.E $m^3 - 8 = 0$

$\Rightarrow m = 2, -1 \pm \sqrt{3}i$

Hence the general sol$^n$ of ④ is

$$x = c_1 e^{2t} + e^{-t}(c_2 \cos\sqrt{3}t + c_3 \sin\sqrt{3}t)$$

Where $c_1, c_2, c_3$ being arbitrary constants

Now from ① we get $y = \frac{1}{2}\frac{dx}{dt}$

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$\therefore$ calculating we get

$$y = c_1 e^{2t} + \frac{e^{-t}}{2}\left[(\sqrt{3}c_3 - c_2)\cos\sqrt{3}t - (c_3 + \sqrt{3}c_2)\sin\sqrt{3}t\right] \quad \text{—(6)}$$

Also from (2) we get

$$z = \frac{1}{2}\frac{dy}{dt}$$

$$= c_1 e^{2t} + \frac{e^{-t}}{2}\left[(c_2 + \sqrt{3}c_3)\cos\sqrt{3}t + (\sqrt{3}c_2 - c_3)\sin\sqrt{3}t\right] \quad \text{—(7)}$$

(5), (6) and (7) constitute the solution of the given equations.

Solve

$$\frac{dx}{dt} = y+z$$

$$\frac{dy}{dt} = z+x$$

$$\frac{dz}{dt} = x+y$$

Solⁿ:- Here

$$\frac{dx}{dt} = y+z \quad \text{—(1)}$$

$$\frac{dy}{dt} = z+x \quad \text{—(2)}$$

$$\frac{dz}{dt} = x+y \quad \text{—(3)}$$

Differentiating (1) w.r.to $t$ we get

$$\frac{d^2x}{dt^2} = \frac{dy}{dt} + \frac{dz}{dt}$$

$$= z + x + x + y$$

$$\Rightarrow d^2x/dt^2 = y + z + 2x$$

$$\Rightarrow \frac{d^2x}{dt^2} = \frac{dx}{dt} + 2x \quad \left(\text{since } y+z = \frac{dx}{dt}\right)$$

$$\Rightarrow \frac{d^2x}{dt^2} - \frac{dx}{dt} - 2x = 0$$

AE $m^2 - m - 2 = 0$

$\Rightarrow m = 2, -1$

Hence $x = c_1 e^{2t} + c_2 e^{-t}$; ($c_1, c_2$ being arbitrary constants) — (4)

Again by (2) − (3) we get

$$\frac{d}{dt}(y-z) = -(y-z)$$

$\therefore (y-z) = c_3 e^{-t}$ — (5)

$c_3$ being arbitrary consts.

Again from (1) and (4)

$y+z = \frac{dx}{dt} = -c_1 e^{-t} + 2c_2 e^{2t}$ — (6)

From (5) and (6) we get

$y = \frac{1}{2}(c_3 - c_1)e^{-t} + c_2 e^{2t}$

and $z = -\frac{1}{2}(c_1 + c_2)e^{-t} + c_2 e^{2t}$

$\therefore$ the general soln will be

$$\left.\begin{aligned} x &= c_1 e^{-t} + c_2 e^{2t} \\ y &= \tfrac{1}{2}(c_3 - c_1)e^{-t} + c_2 e^{2t} \\ z &= -\tfrac{1}{2}(c_3 + c_1)e^{-t} + c_2 e^{2t} \end{aligned}\right\}$$

$c_1, c_2, c_3$ being arbitrary constants.